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ON STABILITY OF STEADY MOTIONS OF A DYNAMICALLY SYMMETRIC SOLID BODY AT A TRIANGULAR POINT OF LIBRATION*

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The motion of a dynamically symmetric solid body is considered relatively to its center of mass, placed at the triangular libration point L_4 of the circular restricted problem of three bodies. It is assumed that the motion of the basic bodies M_1 and M_2 of ultimate mass m_1 and m_2, and that of the solid body center of mass O is defined by the equations of the plane circular restricted problem of three bodies. The dimensions of the solid body are assumed small in comparison with the distance of its center of mass to M_1 and M_2, which enables us to neglect the effect of motion of the solid body about its center of mass on the motion of that center itself.

Sufficient conditions of stability of a gyrostat satellite were obtained in /1/ on the assumption that the satellite center of mass is located at the points of libration. The steady motions of the body whose center of mass is located during the whole time of motion at one of the libration points in the gravitational field of two point mass were obtained in /2/. In /3/ the problem of their stability was investigated in the first approximation, and the sufficient conditions of stability of certain of these motions were obtained in /4/. The investigation of stability of the relative equilibrium of an axisymmetric solid body whose center of mass moves along the periodic orbit of the circular restricted problem of three bodies was carried out in /5/.

To investigate the solid body motion relative to its center of mass we introduce two systems of coordinates: the orbital $OXYZ$ (the axis OZ is a continuation of the radius-vector M_1L_4, the axis OY is normal to the plane of triangle $M_1M_2L_4$ and is directed so that viewed from its end the rotation of points M_1 and M_2 is counterclockwise, the OX axis complements the axes OY and OZ to a right-hand trihedral), and the attached system $Oxyz$ (whose axes are directed along the principal central axes of inertia of the body, with the Oz axis directed along its axis of dynamic symmetry). The orientation of the attached coordinate system relative to the orbital one is defined by Euler's angles ψ, θ, φ.

From the expression for the body kinetic energy, for the projections p, q, r of absolute angular velocity of the body on axes Ox, Oy, Oz, and for the force function /6/ it follows that φ is a cyclic coordinate, hence the projection of the absolute angular velocity on the Oz axis is constant $r = r_0 = \mathrm{const}$.

We select the distance between points M_1 and M_2 as the unit of length. Then (n is the angular velocity of bodies M_1 and M_2, and f is the universal gravitational constant)

$$k_1 = (1-\mu)n^2, \quad k_2 = \mu n^2$$
$$(\mu = m_2/(m_1+m_2), \quad k_i = fm_i)$$

Assuming in the Lagrange equations of motion of the body axis of symmetry relative to the orbital system of coordinates to be $\psi'' = \psi' = \theta'' = \theta' = 0,\ \psi = \psi_0,\ \theta = \theta_0$ (the prime denotes differentiation with respect to $\tau = nt$), we obtain for the determination of steady motions of the body the following system of equations:

$$\begin{gathered} 4\sin 2\psi_0 \sin^2\theta_0 + 8\alpha\beta \sin\psi_0 \sin\theta_0 - \\ 3(\alpha-1)\mu(3\sin 2\psi_0 \sin^2\theta_0 + \sqrt{3}\cos\psi_0 \sin 2\theta_0) = 0 \\ 4\cos^2\psi_0 \sin 2\theta_0 + 8\alpha\beta\cos\psi_0\cos\theta_0 - 3(\alpha-1)[4(1-\mu)\sin 2\theta_0 + \\ \mu(\sin 2\theta_0 - 3\sin^2\psi_0 \sin 2\theta_0 - 2\sqrt{3}\sin\psi_0\cos 2\theta_0)] = 0 \\ \alpha = C/A,\ \beta = r_0/n \end{gathered} \tag{1}$$

Omitting the complete analysis of system (1) for arbitrary parameters α, β, μ, we shall consider the following of its particular solution:

$$\theta_0 = \pi/2, \quad \psi_0 = \pi \quad (\beta - \text{is any}) \tag{2}$$

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$$\theta_0 = \frac{1}{2}\,\mathrm{arc\,tg}\,\frac{\sqrt{3}\mu}{2-3\mu}, \quad \psi_0 = \frac{\pi}{2}; \quad \theta_0 = \frac{1}{2}\,\mathrm{arc\,tg}\,\frac{\sqrt{3}\mu}{2-3\mu} + \frac{\pi}{2}, \tag{3}$$

$$\psi_0 = \frac{\pi}{2} \quad (\beta = 0)$$

Solution (2) and, also (3), were obtained in /2/ for $\mu = 0.5$

For solution (2) the axis of dynamic symmetry of the body is normal to the plane of triangle $M_1M_2L_4$ and the body rotates about the axis with constant velocity φ. For solution (3) the axis of dynamic symmetry of the body lies in the plane of the triangle $M_1M_2L_4$ under angle θ_0 to the radius-vector M_1O, and the angular velocity of proper rotation $\varphi^{\cdot}$, as well as r_0 are equal zero.

To investigate the stability of obtained solutions we use the equation of motion in the Hamiltonian form.

Motion (2) corresponds to the solution of Hamilton equations

$$\theta = \pi/2 + x_1,\ p_\theta = y_1,\ \psi = \pi + x_2,\ p_\psi = y_2 \tag{4}$$

We expand the Hamiltonian function in series in the neighborhood of solutions (4), setting

$$\theta = \pi/2 + x_1,\ p_\theta = y_1,\ \psi = \pi + x_2,\ p_\psi = y_2$$

We obtain

$$H = H_2 + H_4 + \dots \tag{5}$$

$$H_2 = \frac{1}{2}(y_1^2 + y_2^2) + \left[\frac{\alpha^2\beta^2 - \alpha\beta}{2} + \frac{3}{2}(\alpha - 1)\left(1 - \frac{3}{4}\mu\right)\right] x_1^2 +$$
$$\frac{3\sqrt{3}}{4}(\alpha - 1)\mu x_1 x_2 + \left[\frac{\alpha\beta}{2} + \frac{9}{8}(\alpha - 1)\mu\right] x_2^2 + (\alpha\beta - 1) x_1 y_2 + x_2 y_1$$

$$H_4 = \left[\frac{1}{3}\alpha^2\beta^2 - \frac{5}{24}\alpha\beta - \frac{1}{2}(\alpha - 1)\left(1 - \frac{3}{4}\mu\right)\right] x_1^4 +$$
$$\frac{1}{8}[2\alpha\beta - 9(\alpha - 1)\mu] x_1^2 x_2^2 - \left[\frac{1}{24}\alpha\beta + \frac{3}{8}(\alpha - 1)\mu\right] x_2^4 -$$
$$\frac{\sqrt{3}}{8}(\alpha - 1)\mu x_1 x_2^3 - \frac{\sqrt{3}}{2}(\alpha - 1)\mu x_1^3 x_2 + \frac{1}{2} x_1 x_2^2 y_2 +$$
$$\left(\frac{5}{6}\alpha\beta - \frac{1}{3}\right) x_1^3 y_2 - \frac{1}{6} x_2^3 y_1 + \frac{1}{2} x_1^2 y_2^2$$

The characteristic equation of the linear system defined by the form H_2, is of the form

$$\lambda^4 + [(\alpha\beta - 1)^2 + 3\alpha - 2]\lambda^2 + (\alpha\beta - 1)(\alpha\beta + 3\alpha - 4) + \frac{27}{4}(\alpha - 1)^2\mu(1 - \mu) = 0 \tag{6}$$

For stability it is necessary that all roots of this equation be pure imaginary. The sufficient condition of stability is the condition of positive definiteness of the quadratic form H_2 /7/.

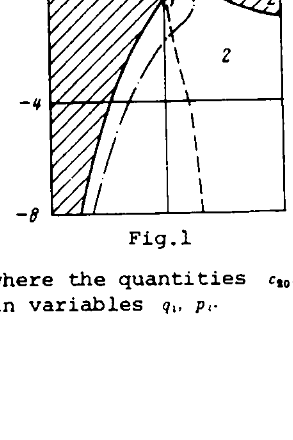

Fig.1

In Fig.1 for $\mu = 0.01215$, which corresponds to the system Earth – Moon, in the parameter plane α, β $(0 < \alpha < 2,\ -\infty < \beta < +\infty)$ the motion in the shaded region is unstable, in region 1 it is stable, and in region 2 only the necessary conditions of stability are satisfied. In region 2 solution (2) is in the first approximation stable. In that region the form H_2 is not of fixed sign but the characteristic equation (6) has only pure imaginary roots.

To solve the problem of stability in region 2 in the strictly nonlinear sense by means of real canonical transformation $x_i, y_i \to q_i, p_i$, obtained in /8/, we reduce function H_2 to the normal form

$$H_2 = \frac{1}{2}\omega_1(q_1^2 + p_1^2) - \frac{1}{2}\omega_2(q_2^2 + p_2^2)$$

and then (since $H_3 \equiv 0$) by the Birkhoff transformation $q_i, p_i \to q_i^*, p_i^*$ we reduce the Hamiltonian H to the form

$$H = \omega_1 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + \tag{7}$$
$$a r_2 \sqrt{r_1 r_2}\sin(\varphi_1 + 3\varphi_2) + b r_2 \sqrt{r_1 r_2}\cos(\varphi_1 + 3\varphi_2)$$
$$q_i^* = \sqrt{2r_i}\sin\varphi_i,\ p_i^* = \sqrt{2r_i}\cos\varphi_i$$

where the quantities c_{20}, c_{11}, c_{02}, and a and b are calculated using the coefficients of form H_4 in variables q_i, p_i.

If the system does not have fourth order resonance $\omega_1 = 3\omega_2$, then the last two terms in formula (7) are absent. In that case the Arnol'd — Mozer theorem, the equilibrium position of system with the Hamiltonian (5) is stable, if $D(\alpha, \beta) = c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 \neq 0$. In Fig.1 the curve $D(\alpha, \beta) = 0$ for $\mu = 0.01215$ is shown by the dash line. The question of its stability was not considered.

Along the resonance curve $\omega_1 = 3\omega_2$ (shown by the dash-dot line in Fig.1) according to Markeev's theorem /8/, the equilibrium position is stable, if

$$3\sqrt{3}\sqrt{a^2+b^2} < |c_{20} + 3c_{11} + 9c_{02}|$$

and unstable when the inequality sign is the opposite.

Computer calculations have shown that on the resonance curve in region 2 contains two sections of instability: $(\mu = 0.01215)$: $-1.747 < \beta < -1.573$ and $0.386 < \beta < 0.449$.

Let us now investigate the stability of the first of solutions (3). The analysis of the second solution is analogous. The following solution of Hamilton equations:

$$\psi_0 = \frac{\pi}{2}, \quad \theta_0 = \frac{1}{2} \operatorname{arc\,tg} \frac{\sqrt{3}\mu}{2-3\mu}, \quad p_\psi = 0, \quad p_\theta = 1 \tag{8}$$

corresponds to the considered here motion.

We introduce new canonical variables x_i, y_i using formulas

$$\theta = \theta_0 + x_1, \; p_\theta = 1 + y_1, \; \psi = \pi/2 + x_2, \; p_\psi = y_2$$

The expression for H_2 is

$$H_2 = \frac{1}{2}\left[\frac{y_2^2}{\sin^2\theta_0} + y_1^2 + 2x_2y_2 \operatorname{ctg}\theta_0 + x_2^2\right] - \tag{9}$$
$$\frac{3}{4}(\alpha - 1)[2(1-\mu)\cos 2\theta_0 - \mu\cos 2\theta_0 + \sqrt{3}\mu\sin 2\theta_0]\,x_1^2 -$$
$$\frac{3}{16}(\alpha - 1)\mu(6\sin^2\theta_0 + \sqrt{3}\sin 2\theta_0)\,x_2^2$$

It can be shown that the inequality

$$\alpha < 1 \tag{10}$$

is a condition for the roots of characteristic equation of the linear system to be pure imaginary, as well as the condition of positive definiteness of the quadratic form (9), i.e. (10) is the necessary and sufficient condition of stability of solution (8). Condition (10) means that the body motion is stable, when its polar moment of inertia is smaller that its equatorial moment, i.e. the body is elongated along its axis of symmetry, a condition obtained in /4/. For the second solution of (3) the stability condition is $\alpha > 1$.

We note in conclusion that the considered here problem is a natural extension of the well studied problem of regular precession of a satellite in circular orbit. A bibliography of that problem appeared in /6/. The results of the present investigation pass when $\mu = 0$ to the respective results of /9,10/.

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